1. Suppose $A = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} 2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this problem you must answer using the methods requested (In particular you do not have to calculate A or A^{-1} at any point). No credit will be given for other methods.

(a) For a vector $\mathbf{b} \in \mathbb{R}^4$, could the equation $A\mathbf{x} = \mathbf{b}$ have 0, 1, or ∞ solutions? Explain how you know by looking at L and U.

(b) Find all solutions to the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (2, 0, -6, 6)$ using L and U with forward and back substitution.

(c) Use the Gauss-Jordan algorithm to find L^{-1} and U^{-1} or to show they don't exist.

(d) Write an expression for A^{-1} as the product of two matrices (do not compute!).

(e) Write an expression for A^T as the product of two matrices (do not compute!).

(f) Write L as a product of elimination matrices. [Hint: Use Part 1c and take inverses.]

Solution:

(a) For a vector $\mathbf{b} \in \mathbb{R}^4$, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution. This is because A has 4 pivots, which are found on the diagonal of U. Equivalently, since all diagonal entries of both L and U are nonzero, every step in the forward and back substitution used to solve $A\mathbf{x} = \mathbf{b}$ will have exactly one solution.

(b) The equation $A\mathbf{x} = \mathbf{b}$ is equivalent to $LU\mathbf{x} = (1, 0, -3, 3)$. We first solve $L\mathbf{c} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -6 \\ 6 \end{pmatrix}$$

Forward substitution implies $c_1 = 2$, $c_2 = 0$, $c_3 = 6$ and $c_4 = 6$.
Next we solve solve $U\mathbf{x} = \mathbf{c}$:

$$\begin{pmatrix} 2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \\ 6 \end{pmatrix}$$

Back substitution implies $x_4 = 6$, $x_3 = 2$, $x_2 = 0$ and $x_1 = -2$.
Thus the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (-2, 0, 2, 6)$.

(c) First we do the Gauss-Jordan algorithm beginning with $[L|I]$:

$$[L|I] = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

We clear out the first column by subtracting 3(Row 1) from Row 4 to obtain

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

We clear out the second column by subtracting (1/2)(Row 2) from Row 3 to obtain

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

We clear out the third column by subtracting Row 3 from Row 4 to obtain

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1/2 & -1 & 1 \end{array}\right)$$

Finally we multiply Row 2 by 1/2 and Row 3 by -1 to get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1/2 & -1 & 1 \end{array}\right)$$

Thus

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & 0 \\ -3 & 1/2 & -1 & 1 \end{pmatrix}.$$

Next we do the Gauss-Jordan algorithm beginning with $[U|I]$:

$$[U|I] = \left(\begin{array}{cccc|cccc} 2 & 4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The first column is already clear, so we clear out the second column by subtracting 4(Row 2) from Row 1 to obtain

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 1 & 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The third column is already clear, so we clear out the fourth column by subtracting 2(Row 4) from Row 3 and subtracting Row 4 from Row 1 to obtain

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & -4 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Finally we multiply Row 1 by $1/2$ and Row 3 by $-1/3$ to get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & -2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/3 & 2/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Thus

$$U^{-1} = \begin{pmatrix} 1/2 & -2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note the inverse of a lower triangular matrix is lower triangular, and the inverse of an upper triangular matrix is upper triangular.

(d) $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 1/2 & -2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & 0 \\ -3 & 1/2 & -1 & 1 \end{pmatrix}.$

(e) $A^T = (LU)^T = U^T L^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

(f) The elimination matrices corresponding to the steps of the Gauss-Jordan algorithm in Part 1c give

$E_3E_2E_{43}E_{32}E_{41}L =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} L$$

$= I_4$

Thus

$$L = E_{41}^{-1}E_{32}^{-1}E_{43}^{-1}E_{2}^{-1}E_{3}^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix}$.

   (a) Find the RREF for A.

   (b) Find the rank of A and the dimensions of each of the four fundamental subspaces for A.

   (c) Find bases for the column space, the row space, and the null space of A.

   (d) Find all solutions to $A\mathbf{x} = (1, 4, 5, 3)$ or explain why none exists.

   (e) Find all solutions to $A\mathbf{x} = (1, 4, 6, 3)$ or explain why none exists.

   (f) Show that the columns of A are not linearly independent.

   (g) Give a formula for the projection of $(3, 0, 3, 1)$ onto $C(A)$. (You do not need to compute, just write a formula.)

Solution:

(a) Beginning with $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix}$,

We have a pivot (1) in Row 1 Column 1, so we subtract Row 1 from Row 2 and subtract 2(Row 1) from Row 3 to get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The next pivot (-1) in Row 2 Column 2, so we subtract Row 2 from Rows 3 and 4 and add Row 2 to Row 1 to get

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we multiply Row 2 by -1 to obtain the RREF:

$$R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Since there are two pivots, the rank of A is 2.

- The dimension of the row space is equal to 2, the rank.
- The dimension of the column space is equal to 2, the rank.
- By the Rank-nullity theorem, the dimension of the null space is $n - r = 3 - 2 = 1$.
- By the Rank-nullity theorem, the dimension of the left null space is $m - r = 4 - 2 = 2$.

(c) A basis for $C(R)$ is given by the first and second columns, and since row operations preserve dependence relations among columns, a basis for $C(A)$ is given by the first and second columns of A, $\{(1, 1, 2, 0), (1, 0, 1, -1)\}$.

Since row operations preserve linear independence among sets of rows and rows 1 and 2 are a basis for the row space of R, a basis for $C(A^T)$ is given by the first and second rows of A, $\{(1, 1, 1), (1, 0, 2)\}$.

A basis for the null space is given by the columns of $N = \begin{pmatrix} -F \\ I_1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$,

i.e. a basis for the null space is $\{(-2, 1, 1)\}$.

(d) Performing the same row operations on $\mathbf{b} = (1, 4, 5, 3)$ as we did on A (R2-R1, R3-2R1, R1+R2, R3-R2, R4 - R2, (-1)R2), we get that the system $A\mathbf{x} = \mathbf{b}$ has the same solutions as $R\mathbf{x} = (4, -3, 0, 0)$, i.e.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix}.$$

Setting the free variable $x_3 = 0$, we obtain a particular solution $\mathbf{x}_p = (4, -3, 0)$. Since any solution is of the form $\mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n \in N(A)$, all solutions are $\{(4, 3, 0) + c(-2, 1, 1) \mid c \in \mathbb{R}\}$.

(e) Performing the same row operations on $\mathbf{b} = (1, 4, 6, 3)$ as we did on A (R2-R1, R3-2R1, R1+R2, R3-R2, R4 - R2, (-1)R2), we get that the system $A\mathbf{x} = \mathbf{b}$ has the same solutions as $R\mathbf{x} = (4, -3, 1, 0)$, i.e.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}.$$

Since the third equation $0x_1 + 0x_2 + 0x_3 = 1$ has no solution, the system has no solutions.

(f) Since $N(A)$ contains the vector $(-2,1,1)$, i.e. $A(-2,1,1) = \mathbf{0}$, we get a nontrivial linear combination of the columns of A equal to $\mathbf{0}$:

$$-2\begin{pmatrix}1\\1\\2\\0\end{pmatrix} + 1\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} + 1\begin{pmatrix}1\\2\\3\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Thus the columns of A are not linearly independent.

(g) Let B be a matrix whose columns are a basis for $C(A)$, e.g.

$$B = \begin{pmatrix}1 & 1\\1 & 0\\2 & 1\\0 & -1\end{pmatrix}$$

Then the projection of $(3,0,3,1)$ onto $C(A)$ is given by

$$B(B^TB)^{-1}B^T(3,0,3,1).$$

3. Let m and n be natural numbers, with $1 \leq m \leq n$. Suppose $\mathbf{u} \in \mathbb{R}^n$ and let $U = \{c\mathbf{u} : c \in \mathbb{R}\}$. Let V be a subspace of $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, and suppose that for $1 \leq i \leq m$, $\mathbf{u}$ is orthogonal to $\mathbf{v}_i$.

For each of the following statements, either prove it, or explain why it is not necessarily true.

(a) U is a subspace of $\mathbb{R}^n$.

(b) For all $\mathbf{v} \in V$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

(c) $V^\perp = U$.

(d) $\mathbf{u} \notin V$.

Solution:

(a) We prove U is a subspace of $\mathbb{R}^n$:

- Since $0 \in \mathbb{R}$, $0\mathbf{u} = \mathbf{0} \in U$.
- Let $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then there exist $c, d \in \mathbb{R}$ such that $\mathbf{u}_1 = c\mathbf{u}$ and $\mathbf{u}_2 = d\mathbf{u}$. Thus
$$\mathbf{u}_1 + \mathbf{u}_2 = c\mathbf{u} + d\mathbf{u} = (c+d)\mathbf{u}.$$
Since $(c+d) \in \mathbb{R}$, $\mathbf{u}_1 + \mathbf{u}_2 \in U$, so U is closed under addition.
- Let $\mathbf{u}_1 \in U$ and $a \in \mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that $\mathbf{u}_1 = c\mathbf{u}$. Then
$$a\mathbf{u}_1 = a(c\mathbf{u} = (ac)\mathbf{u}.$$
Since $(ac) \in \mathbb{R}$, $a\mathbf{u}_1 \in U$, so U is closed under scalar multiplication

Since $\mathbf{0} \in U$ and U is closed under addition and scalar multiplication, U is a vector space.

(b) We prove that for all $\mathbf{v} \in V$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$.
Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis for V, $\mathbf{v}$ can be uniquely expressed as a linear combination of these basis vectors,

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m.$$

To prove $\mathbf{u}$ is orthogonal to $\mathbf{v}$, we show that the dot product $\mathbf{u}^T\mathbf{v} = 0$:

$$\begin{aligned}
\mathbf{u}^T\mathbf{v} &= \mathbf{u}^T(c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m) && \text{substitution} \\
&= \mathbf{u}^T(c_1\mathbf{v}_1) + \cdots + \mathbf{u}^T(c_m\mathbf{v}_m) && \text{linearity} \\
&= c_1(\mathbf{u}^T\mathbf{v}_1) + \cdots + c_m(\mathbf{u}^T\mathbf{v}_m) && \text{linearity} \\
&= c_1(0) + \cdots + c_m(0) && \mathbf{u} \perp \mathbf{v}_i \\
&= 0 + \cdots + 0 \\
&= 0
\end{aligned}$$

(c) $V^\perp = U$ is not necessarily true. While $U \subseteq V^\perp$, there could be other vectors that are orthogonal to V that are not in U.
Note that if $m = n - 1$, then the statement $V^\perp = U$ is true.

(d) $\mathbf{u} \notin V$ is not necessarily true. It is false if $\mathbf{u} = \mathbf{0}$. (If $\mathbf{u} \neq \mathbf{0}$, it is true!)